MONOTONOCITY

EXERCISE – I HINTS & SOLUTIONS

Sol.1 **C**

Let $z = x^3$

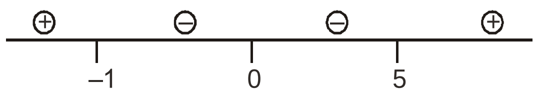
$y = 6x^2 + 15x + 5$

$\frac{dy}{dx} > 1$

$\frac{12x+15}{3x^2} - 1 > 0$

$\frac{12x+15-3x^2}{3x^2} > 0$

$\frac{x^2-4x-5}{x^2} < 0 \Rightarrow \frac{(x+1)(x-5)}{x^2} < 0$



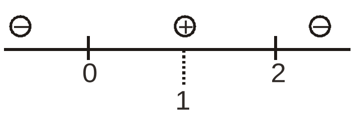
$x \in (-1, 5)$

Sol.2 **C**

$f(x) = \frac{|x-1|}{x^2}$

$x \geq 1 \qquad f(x) = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$

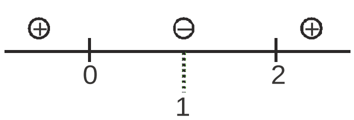
$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = -\frac{(x-2)}{x^3}$



$x \in (2, \infty)$ decreasing

$x < 1$ $f(x) = \frac{1-x}{x^2}$

$f'(x) = \frac{x-2}{x^3}$



$x \in (0, 1)$

$x \in (0, 1) \cup (2, \infty)$ decreasing

Sol.3 **A**

$y = (a + 2)x^3 - 3ax^2 + 9ax - 1 \qquad x \in R$

Case-I when $a = -2$

$y = 6x^2 - 18x - 1$ which is non-monotonic

Case-II when $a \neq -2$

$y = (a + 2)x^3 - 3ax^2 + 9ax - 1$

$y' = 3(a + 2)x^2 - 6ax + 9a < 0 \quad \forall x \in R$

$\boxed{a < -2}$ $D < 0$

$36a^2 - 36.3a(a + 2) < 0$

$a^2 - 3a^2 - 6a < 0$

$2a(a + 3) > 0$

$a > 0, a < -3$

$a \in (-\infty, -3)$

check for end value.

$a = -3$

$f'(x) = -3x^2 + 18x - 27 = -3(x-3)^2 < 0$ always.

$\boxed{a \in (-\infty, -3]}$

Sol.4 **B**

$f(x) = \left(\frac{\sqrt{p+4}}{1-p} - 1\right) x^5 - 3x + \ell n5$

$f'(x) = 5\left(\frac{\sqrt{p+4}}{1-p} - 1\right) x^4 - 3$

f '(x) will have critical point have coefficient of x^4 positive.

But if the coefficient of x^4 negative and we will not get any critical point and f(x) will be decreasing function. so $\frac{\sqrt{p+4}}{1-p} - 1 \leq 0$

If equality holds f '(x) = – 3 is still decreases.

Case-I $1 - p > 0 \Rightarrow p < 1$

$p + 4 \leq (1 - p)^2$

$p^2 - 3p - 3 \geq 0$

$p \in \left(-\infty, \frac{3-\sqrt{21}}{2}\right]$

Case-II $1 - p < 0 \Rightarrow p > 1$

$\frac{\sqrt{p+4}}{1-p} \leq 1$

always true

$p \in (1, \infty)$

$p + 4 \geq 0$

$p \geq -4$

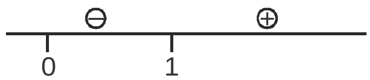
final answer $p \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$

Sol.5 **D**

$f(x) = x\ell nx - x + 1 \qquad D_f : x \in R^+$

$f'(x) = \ell nx + 1 - 1$

$x = 1$ Critical point



If $x \in (0, 1)$ f is ↓ing

$f(1) > f(x) > f(0) \Rightarrow 0 > f(x) > 1$ positive

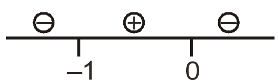
If $x \in (1, \infty)$ & is ↑ing

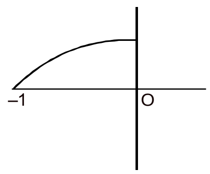
$f(x) > f(1) \Rightarrow \boxed{f(x) > 0}$

Sol.6 **B**

Let $f(x) = \ell n(1 + x) - x \quad D_f : \boxed{x > -1}$

$f'(x) = \frac{1}{1+x} - 1 = \frac{-x}{1+x}$



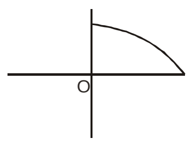


In $x \in (-1, 0)$ f is ↑ing

$f(x) \le f(0)$

$f(x) \le 0$

In $x \in (0, \infty)$ f is ↓ing.



$f(x) \le f(0)$

$f(x) \le 0$

Sol.7 **D**

$y = f(x)$

$f'(x) > 0 \Rightarrow f(x) \uparrow$

$f''(x) < 0 \Rightarrow f(x)$ is concave downward.

Sol.8 **C**

$f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$

$f'(x) = 4x^3 + 3ax^2 + 3x$

$f'(x) = 12x^2 + 6ax + 3$

$= 3(4x^2 + 2ax + 1)$

for concave upward

$f''(x) \ge 0$

$4x^2 + 2ax + 1 \ge 0$

$4a^2 - 16 \le 0$

$a^2 - 4 \le 0$

$-2 < a \le 2$

Sol.9 **D**

$y = ax^3 + bx^2$

(1, 3)

$3 = a + b \qquad ...(1)$

$y' = 3ax^2 + 2bx$

$y'' = 6ax + 2b\,|_{x=1} = 0$

$6a + 2b = 0$

$b = -3a$

$3 = a - 3a$

$a = -\frac{3}{2}, \; b = \frac{9}{2}$

Sol.10 **A**

$f(x) = x^3 - 6x^2 + ax + b$

$f(1) = f(3)$

$1 - 6 + a + b = 27 - 54 + 3a + b$

$\boxed{a = 11}$

$f(1) = 6 + b$

$f(3) = 6 + b$

b can be anything

Sol.11 **C**

$f(x) = x(x + 3)\,e^{-x/2}$

$f(x) = (x^2 + 3x)\,e^{-x/2}$

$f(-3) = 0 = f(0)$ Rolle's theorem. is applicable

$f'(x) = (x^2 + 3x)\,e^{-x/2}\left(-\frac{1}{2}\right) + e^{-x/2}(2x + 3) = 0$

$x = 3, -2$

$x = -2 \in [-3, 0]$

Sol.12 **D**

$(\ell na)\,h(x) = \ell n\, f(x)\, g(x)$

$= \ell n\,[a^{\{a^{|x|}\,\mathrm{sgn}\,x\} + [a^{|x|}\,\mathrm{sgn}\,x]}]$

$= \ell n\; a^{a^{|x|}\,\mathrm{sgn}\,x}$

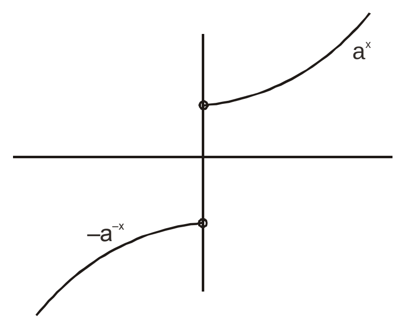
$(\ell n\,a)\,h(x) = a^{|x|}\,\mathrm{sgn}\,x\,(\ell n\,a)$

$h(x) = a^{|x|}\,\mathrm{sgn}\,x$

$h(x) = a^x \qquad x > 0$

$\quad = 0 \qquad x = 0$

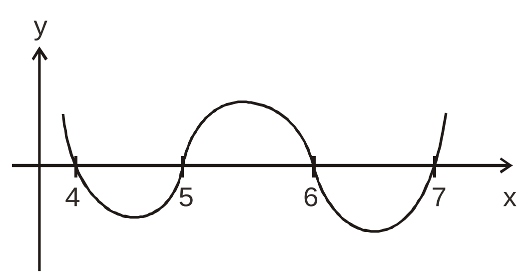
$\quad = -a^{-x} \qquad x < 0$



h is odd and ↑ing.

Sol.13 **B**

$f(x) = (x - 4)(x - 5)(x - 6)(x - 7)$



This curve f(x) cutting, y = 0 curve four times then in between [4, 7] there will be three points where $f'(x) = 0$ by rolle's therm.

Alter:

$f'(x) = (x-5)(x-6)(x-7) + (x-4)(x-6)(x-7) + (x-4)(x-5)(x-7) + (x-4)(x-5)(x-6)$

$f'(4) < 0$

$f'(5) > 0$

$f'(6) < 0$

$f'(7) > 0$

by L.M.V.T. we can say that is [4, 5] one root

Sol.14 C

$f''(x) = (a^2 - 2a - 2) - \sin x > 0 \quad \text{or} < 0$

$a^2 - 2a - 2 > \sin x$ or $a^2 - 2a - 2 < \sin x$

$a^2 - 2a - 2 \ge 1$ or $a^2 - 2a - 2 \le 1$

$a^2 - 2a - 3 \ge 0$ or $a^2 - 2a - 1 \le 0$

$a \ge 3,\ a \le -1$ or $a \in \left[1-\sqrt{2}, 1+\sqrt{2}\right]$

Sol.15 D

Given that $f(x)\, f'(x) < 0$

$f(x)\, f'(x) < 0$

$$y = |f(x)| \begin{cases} f(x) & f(x) > 0 \\ -f(x) & f(x) < 0 \end{cases}$$

$$y' = \begin{cases} f'(x) & f(x) > 0 & ...(1) \\ -f'(x) & f(x) < 0 & ...(2) \end{cases}$$

(1) $f(x) > 0$, $f'(x) < 0$ decreasing

(2) $f(x) < 0$, $f'(x) > 0$, but $y' < 0$

so $|f(x)|$ is decreasing function.

Sol.16 C

$$f'(x) = \frac{1}{2}\left[\frac{(1-\cos x)2x - x^2 \sin x}{(1-\cos x)^2}\right]$$

$$= \frac{x}{2}\left[\frac{2(1-\cos x) - x\sin x}{(1-\cos)^2}\right]$$

$$= 4\sin^2\frac{x}{2} - 2x\sin\frac{x}{2}\cos\frac{x}{2}$$

$$= 2x\sin\frac{x}{2}\cos\frac{x}{2}\left[\frac{\tan\frac{x}{2}}{\frac{x}{2}} - 1\right] > 0$$

$\Rightarrow$ f is increasing

$$g'(x) = \frac{1}{6}\left[\frac{(x-\sin x)2x - x^2(1-\cos x)}{(x-\sin x)^2}\right]$$

$= 2x\cos^2 x - 4\sin x \cos x$

$$= 2x\cos^2 x\left[1 - \frac{\tan\frac{x}{2}}{\frac{x}{2}}\right] < 0 \Rightarrow \text{g is decreasing}$$

Sol.17 B

$f(x) = x^3 - 6ax^2 + 5x$

$f'(x) = 3x^2 - 12ax + 5$

By LMVT

$$f'(x) = \frac{f(2)-f(1)}{2-1}$$

$$f'(x)\Big|_{x=\frac{7}{4}} = f(2) - f(1)$$

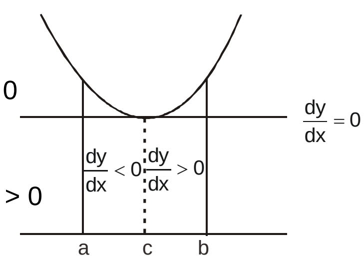
$$3\left(\frac{7}{4}\right)^2 - 12a\left(\frac{7}{4}\right) + 5 = (8 - 24a + 10) - (1 - 6a - 5)$$

$$a = \frac{35}{48}$$

Sol.18 C

For $x \in (a, b)$

$$\frac{dy}{dx}\uparrow \Rightarrow \frac{d^2y}{dx^2} > 0$$



either $\frac{dy}{dx} < 0 \Rightarrow \frac{d^2y}{dx^2} > 0$

$$\frac{dy}{dx} > 0 \Rightarrow \frac{d^2y}{dx^2} > 0$$

Sol.19 A

$f(x) = \log_e x$

$$f'(x) = \frac{f(3)-f(1)}{3-1}$$

$$\frac{1}{c} = \frac{\log_e 3 - \log_e 1}{2}$$

$$c = \frac{2}{\log_e 3} = 2\log_3 e$$

Sol.20 B

$f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{1}{1+(\sin x + \cos x)^2}(\cos x - \sin x) > 0$$

$\cos x - \sin x > 0$

$$\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x > 0$$

$\cos\left(\frac{\pi}{4}+x\right) > 0$

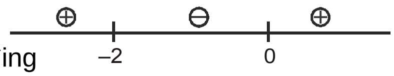
$-\frac{\pi}{2} < \frac{\pi}{4} + x < \frac{\pi}{2} \Rightarrow -\frac{3\pi}{4} < x < \frac{\pi}{4}$

Sol.21 B

$f(x) = x^3 + 6x^2 + 6$

$f'(x) = 3x^2 + 6x = 0$

$x = 0, -2$

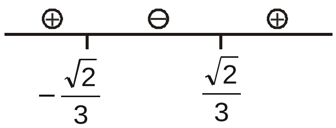


(A) $(-\infty, -4]$ $\uparrow$ing

$f(x) = 3x^3 - 2x + 1$

$f'(x) = 9x^2 - 2 = 0$

$x = \pm\frac{\sqrt{2}}{3}$



$\uparrow$ing is $\left(-\infty, -\frac{\sqrt{2}}{3}\right) \cup \left(\frac{\sqrt{2}}{3}, \infty\right)$

Sol.22 B

$f''(x) = 6(x - 1)$

$f'(x) = 6\left(\frac{x^2}{2} - x\right) + c$

$f'(x) = 3(x^2 - 2x) + c$

$f'(2) = 3$

$3(4 - 4) + c = 3 \Rightarrow c = 3$

$f'(x) = 3(x^2 - 2x) + 3$

$f(x) = 3\left(\frac{x^3}{3} - x^2\right) + 3x + k$

curve passes through (2, 1)

$y = x^3 - 3x^2 + 3x + k$

$1 = 8 - 12 + 6 + k \Rightarrow k = -1$

$y = x^3 - 3x^2 + 3x - 1$

$y = (x - 1)^3$

Sol.23 D

$f(x) = \frac{a\sin x + b\cos x}{c\sin x + d\cos x}$

$f'(x) = \frac{ad\cos^2 x + ad\sin^2 x - bc\sin^2 x - bc\cos^2 x}{(c\sin x + d\cos x)^2}$

$f'(x) = \frac{ad - bc}{(c\sin x + d\cos x)^2} \geq 0 \Rightarrow ad \geq bc$

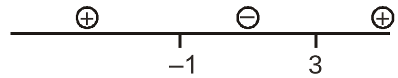
Sol.24 B

$f(x) = x^3 - 3x^2 - 9x + 20$

$f'(x) = 3x^2 - 6x - 9$

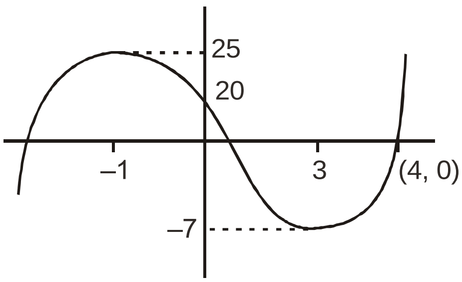
$= 3(x^2 - 2x - 3)$

$= 3(x - 3)(x + 1)$



$f(-1) = 25$ $\quad$ $f(0) = 20$

$f(3) = -7$



check for x = 4

$f(4) = 64 - 48 - 36 + 20 = 0$

f(x) is +ve for x > 4

Sol.25 A

$f(x) = x^2 - x\sin x$

$f'(x) = 2x - x\cos x - \sin x$

$= x(2 - \cos x) - \sin x$

is $\left[0, \frac{\pi}{2}\right]$ $\quad$ $(2 - \cos x)$ is +ve and

sin x is +ve. and

$(2 - \cos x)$ is greater than $\sin x$ so

$f'(x) > 0$

f(x) $\uparrow$ing in $\left[0, \frac{\pi}{2}\right]$

Sol.26 B

$f(x) = (x - 1)(x - 2)(x - 3),\ x \in (0, 4)$

$f'(c) = \frac{f(4) - f(0)}{4 - 0}$

$(c-1)(c-2) + (c-2)(c-3) + (c-3)(c-1) = \frac{6+6}{4}$

$3c^2 - 12c + 8 = 0$

$(3c - 4)(c - 2) = 0$

$c = \frac{4}{3}, 2$

Sol.27 D

let $f(x) = xe^x$

$f'(x) = (x + 1)e^x$

Sol.28 D

$$h(x) = f(x) - g(x) = 1 + x\ \ln\left(x+\sqrt{x^2+1}\right) - \sqrt{x^2+1}$$

$$h'(x) = \ln\left(x+\sqrt{x^2+1}\right) + \frac{x}{\left(x+\sqrt{x^2+1}\right)}$$

$$\frac{(1+x)}{\sqrt{x^2+1}} - \frac{x}{\sqrt{x^2+1}}$$

$$\ln\left(x+\sqrt{x^2+1}\right) = 0 \Rightarrow x = 0$$

h(x) is positive
$h(x) \geq h(0)$
$h(x) \geq 0$
$f(x) \geq g(x)$

Sol.29 C

$f(x) = 8ax - a \sin 6x - 7x - \sin x$
$f'(x) = 8a - 6a \cos 6x - 7 - 5 \cos 5x$
for $a = 0$ $\quad f'(x) < 0$
for $a = -6$ $\quad f'(x) < 0$
for $a = 6$ $\quad f'(x) > 0$

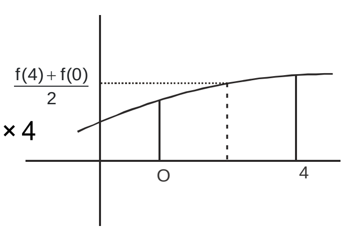
Sol.30 B

$f(0) = 2, g(0) = 0, f(1) = 6, g(1) = 2$
(A) $h(x) = f'(x) - g(x)$
$h(0) = f(0) - g(0) = 2$ wrong
$h(1) = f(1) - g(1) = 6 - 2 = 4$
(B) $h(x) = f(x) - 2g(x)$
$h(0) = f(0) - 2g(0) = 2$ right`
$h(1) = f(1) - 2g(1) = 6 - 4 = 2$

Sol.31 A

$f^2(4) - f^2(0)$

$$\left[\frac{f(4)-f(0)}{4-0}\right] [f(4)+f(0)] \times 4$$



By LMVT

$$f'(a)\left[\frac{f(4)+f(0)}{2}\right] \times 8$$

$f'(a).f(b) \times 8 = 8 f'(a) f(b)$ or $8f'(b) f(a)$

Sol.32 B

$$\int_0^1 (3x^2+4ax+b)\,dx = 0 \quad \text{By LMVT}$$

$x^3 + 2ax^2 + bx \Big|_0^1 = 0 \quad \Rightarrow 1 + 2a + b = 0$

Sol.33 B

$f(x) = \tan x$

$$f'(C) = \sec^2 C = \frac{f(b)-f(a)}{b-a}$$

$$\sec^2 C = \frac{\tan b - \tan a}{b-a} \qquad \sec^2 C \geq 1$$

for $\left[0, \frac{\pi}{2}\right]$

$$f(a, b) = \frac{\tan b - \tan a}{b-a} \geq 1$$

Sol.34 B

$f'(x) = 4ax^3 + 3bx^2 + 2x + 1$
$f'(x) = 12ax^2 + 6bx + 2$
$D = 36b^2 - 96 < 0$ (given)
$f''(x) > 0 \Rightarrow f'(x)$ is increasing

Sol.35 C

(A) $f(0) = 0$
$f(1) = 0$ Rolle's Thrm. is applicable

$$\text{(C) } f'(c) = \frac{f(3)-f(-3)}{3+3}$$

$$e^c = \frac{e^3 - e^{-3}}{6}$$

$$e^c = \frac{e^6 - 1}{6e^3}$$

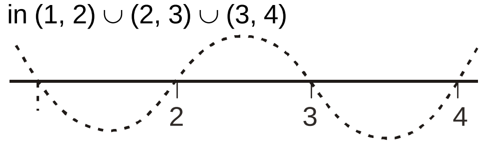
$$c = \ell n\left(\frac{e^6-1}{6e^3}\right) = \ell n\,(e^6 - 1) - \ell n 6 - 3$$

Sol.36 D

(A) LMVT
(B) LMVT
(C) $f(0) = -2$
$f(1) = 4 - 5 + 1 - 2 = -2$
Not applicable

Sol.37 D

$f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$
Roots of $f'(x)$ will lie
in $(1, 2) \cup (2, 3) \cup (3, 4)$



Sol.38 A

$f(x) = 1 + x^m (x - 1)^n$
$f'(x) = = mx^{m-1}(x - 1)^n + nx^m (x - 1)^{n-1} = 0$

$$\frac{m}{x} + \frac{n}{(x-1)} = 0$$

$$mx - m + nn = 0 \qquad \Rightarrow x = \frac{m}{m+n}$$